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TENTAMEN
KWANTUMFYSICA 1
8 april 2004
9:00 – 12:00 uur
Hoor- en werkcollegedocenten:
prof.dr. L.P. Kok
Drs. M. Eenink
Drs. D. Westra.

(10 ptn.) **Opgave 0**

Elke opgave (1, 2, 3, 4) op een apart vel a.u.b. Zet je naam op elk vel met je oplossingen. Zet op vel 1 bovendien duidelijk je studentnummer, adres, geboortedatum, studierichting, en jaar van aankomst. N.B.: Let op de puntenwaardering: die kan verschillen per vraagstuk!

(15 ptn.) **Opgave 1**

(a) The Hermitian operator $\hat{A}$ has an eigenstate $|a\rangle$ with eigenvalue a. If $\hat{B}$ is the inverse operator of $\hat{A}$, prove that $|a\rangle$ is an eigenstate of $\hat{B}$ with eigenvalue a^{-1}.

(b) Compute

$$\int_{-\infty}^{\infty} (x^2 + 5x - 3)\delta(3x - 6)dx. \tag{1.1}$$

(c) Evaluate the commutator $[\hat{H}, \hat{x}]$, where $\hat{H} = \hat{p}^2/2m + \frac{1}{2}m\omega^2\hat{x}^2$.

(25 ptn.) **Opgave 2**

A particle of mass m is subject to the harmonic-oscillator potential. At time $t = 0$ it is described by the following coherent superposition the the corresponding stationary states,

$$\Psi(x, 0) = \frac{1}{4}\varphi_0(x) - \frac{i}{2}\varphi_1(x) + \frac{\sqrt{11}}{4}\varphi_2(x). \tag{2.1}$$

where $\varphi_n(x)$ are the *real* solutions of the time-independent Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{2}m\omega^2 x^2\right]\varphi_n(x) = \hbar\omega(n + \frac{1}{2})\varphi_n(x), n = 0, 1, 2, \ldots. \tag{2.2}$$

(a) Show that the wave function (2.1) is normalized properly.

(b) Determine the wave function at time $t > 0$.

(c) Compute the probability to measure the energy $\frac{3}{2}\hbar\omega$.

(d) Argue (but be brief!) why the functions $\varphi_n(x)$ are orthonormal.

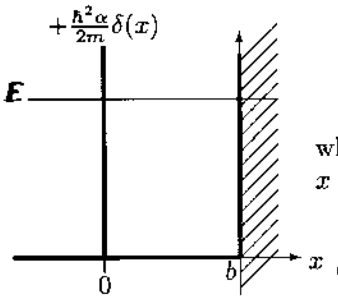
(e) What is the expectation value of the energy?

(f) Prove that the expectation value of x in the state (2.1) equals 0.
[Hint 1: which of the *real* functions $\varphi_n(x)$ are even, and which of the $\varphi_n(x)$ are odd?]
[Hint 2: Remember that $|\Psi(x,0)|^2 = \mathrm{Re}\Psi(x,0)^2 + \mathrm{Im}\Psi(x,0)^2$.]

(25 ptn.)

Opgave 3

A particle of mass m moves in the potential



$$V(x) = \begin{cases} \infty & x > b, \\ \dfrac{\hbar^2\alpha}{2m}\delta(x) & x < b, \end{cases} \tag{3.1}$$

where $b > 0, \alpha > 0$. Consider the situation where the particle approaches the potential from $x = -\infty$ with energy $E > 0$, encounters the potential, and is scattered.

(a) Write down the time-independent Schrödinger equation for the wave function $\psi(x)$ in the region $x < b$.

(b) Verify that the solution to this equation can be written as

$$\psi(x) = \begin{cases} A\exp(ikx) + B\exp(-ikx) & x < 0, \\ C\sin[k(x-b)] + D\cos[k(x-b)] & 0 < x < b, \end{cases} \tag{3.2}$$

and derive an expression for k in terms of E.

(c) Formulate the boundary condition(s) on $\psi(x)$ at $x = b$. What is the consequence of this for the constants C and/or D?

(d) Formulate the boundary condition(s) on $\psi(x)$ and its derivative (!!) at $x = 0$. With the help of this, express B in terms of A.

(e) Compute the reflection coefficient $R \equiv |B|^2/|A|^2$.

(25 ptn.)

Opgave 4

A two-state quantum system has two states: $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. These two states are not eigenstates of the Hamiltonian, as we can see from the Hamiltonian matrix:

$$\hat{H} = \begin{pmatrix} \varepsilon & v \\ v & \varepsilon \end{pmatrix}. \tag{4.1}$$

where ε and v are real, and in fact $v \neq 0$.

(a) The most general state is a normalized linear combination:

$$|\Psi\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Write down the normalization condition on a and b.

(b) Determine the eigenvalues and (normalized) eigenvectors of this Hamiltonian.

(c) The system is initially ($t = 0$) in state $|1\rangle$. What is the state at time t? From this, determine the probability to find the system in state $|2\rangle$ at a later time t.